Proof Systems

A branch of modern mathematics studies the theory of proofs, and this is of interest to philosophers. The main aim of mathematicians is to make proof as secure as possible, but philosophers are interested in how the logic works, and what it reveals about our most reliable forms of reasoning. If the minimal components of proof can be identified, then the process can be both clarified and explained. The most important step in understanding proofs was the creation of a suitable logic, covering both sentences, and their internal structure. Propositional Logic showed how steps could be made from one formula to another, using 'not', 'and', 'or' etc. Then Predicate Calculus showed how the features of particular items, or of every item within a domain (by the use of variables and quantification) could also offer reliable steps through proofs. With the further new background of Set Theory, to map out the underlying structure of mathematics (in the form of 'models'), a full set of tools became available to explore the mechanics of proof.

The idea of an '**axiom**' was the most important idea in the early stages of this exploration. Since earliest times the standard mode of proof in geometry was to start from a set of simple self-evident truths about the topic, with logical steps made by a combination of algebra and intuitive understanding of diagrams. The modern era of proof theory began in the nineteenth century, with the realisation that axioms are not set in stone, and that the axiom defining parallel lines in geometry could be changed in a number of ways. Almost overnight, the authority of axioms as self-evident truths collapsed, and new foundations were needed.

New more abstract axioms were eventually formulated for geometry, and the axiom approach was pursued in the new logics. Just as there were hopes for intuitive geometrical truths, there seemed to be certain clear **logical truths** which might serve as axioms for logic. Thus a proof lays out these indisputable tautologies, and uses the new settled 'classical' logic to build proofs in clear and reliable steps. There was the difficulty, though, of settling on which logical truths to start from, and the restriction of being tied to these axioms, instead of exploring assumptions.

The attention was therefore shifted to **rules**, rather than axioms, and the '**natural deduction**' account of proof in logic emerged, which is the preferred modern view. Given the set of logical connectives (of which there are eight or nine, reducible to 'not' and 'and'), each one of them is assigned an **introduction** rule (I) and an **elimination** rule (E), meaning a complex proof can be built up, and reduced again to simplicity. It is worth summarising them, given here in English (though philosophers should also learn the symbolic forms).

Assumption: **I**) you may assume P; **E**) you can drop an assumption if the proof no longer relies on it

And: **I**) given P and Q, you may combine them into P-and-Q: **E**) given P-and-Q, you can derive P and derive Q

Or: **I**) given P you can derive P-or-Q; **E**) given P-or-Q, if P proves R and Q proves R, derive R and drop P-or-Q

Not: **I**) if P proves Q and not-Q, you may introduce not-Q; **E**) if P and not-P are given, derive anything you like!

If-Then (→): **I**) if P is given and Q then proved, introduce P→Q; **E**) given P→Q and P, derive Q

Double Negation: **I**) given P, introduce not-not-P; **E**) given not-not-P, derive P

Reasoning from assumptions is now permitted. For example, if you assume A, and then use the logic to prove B, you can then state the truth that A implies B ('A→B') whatever A is, and thus '**discharge**' your original assumption of A. If the proofs can free themselves of assumptions, they acquire the ideal of purely logical authority. Natural deduction rules in classical logic can be given in several forms, but the whole system can be presented in this way, with the logical truths now seen as by-products of the rules. The attraction is that natural deduction shows how one truth can be derived from another, but it is 'topic-neutral', meaning that the content of the truths is irrelevant (and that we don't even need truth, since the rules can be applied to falsehoods, which may even produce false but logically valid results). Thus we give the syntax of the logic, and can add an interpretation later (perhaps as truth tables).

Proofs typically list a series of formulae, manipulated according to the rules, usually terminating in a revealing contradiction. A different approach now introduced is a **sequent calculus**. Rather than a formula, a sequent is a statement of what proves what (where, if Γ is a set of formulae, and φ the formula they prove, we write '$\Gamma \vdash \varphi$'). Every line of the proof is a sequent, showing both the result, and what proves the result. This makes proofs even clearer, with every stage of the reasoning on display.

The E-rule for 'or' points to '**proof by cases**', in which it is agreed that a theory has a specific number of options, and each of them in turn is then proved. The **tableaux** proof system extends this approach, creating a branching structure in which options are pursued to conclusions, gradually eliminating the options. The negation of the starting formula is proved if every branch of the tree is 'closed', usually by meeting a contradiction. Tableaux proofs are a more semantic approach, since each branch represents a hypothesis that is true or false. Philosophers like tableaux proofs, because the branches where a proof fails point to counterexamples, which can thus be made more precise in arguments.

In predicate logic there are 'atomic' formulae, of the form 'Fa', meaning that F is predicated of a. The connectives are all truth-functional, and this means that complex formulae can all be tracked back to their atomic ingredients. Hence there is the method of proof 'on the complexity' of the formula. **Inductive** proof starts with an atomic 'base case' and an assumption about further cases, thereby building up to all cases, and **recursion on complexity** works the other way, breaking the complex formula down until the secure atomic components are reached.

The study of proof theory was initiated in an optimistic mood, hoping to secure the whole of mathematics and logic, but once syntax and semantics were distinguished a potential gap opened up. A system is 'complete' if all of its truths can be proved, but it gradually emerged that arithmetic and second-order logic were not complete, so that proof couldn't do the whole job. Thus we become interested in the limits of proof, and why we should believe anything that is beyond proof. Intuitionistic logic is notable for insisting that proof is required, but many inferences seem to be unprovable and yet self-evident.